BOUNDARIES AND LINES: THE ART OF RECONSTRUCTION

E/Ea (they/them)[1]
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SUM-C SUMMER SEMINAR

NSERC
CRSNG

ROADMAP

Introduction to Inverse Problems

CT Imaging and the Radon Transform

Statement of the problem

Probing microlocal boundary properties

Summary

WHAT IS AN INVERSE PROBLEM?

"A problem with the goal of reconstructing information through indirect measurements"

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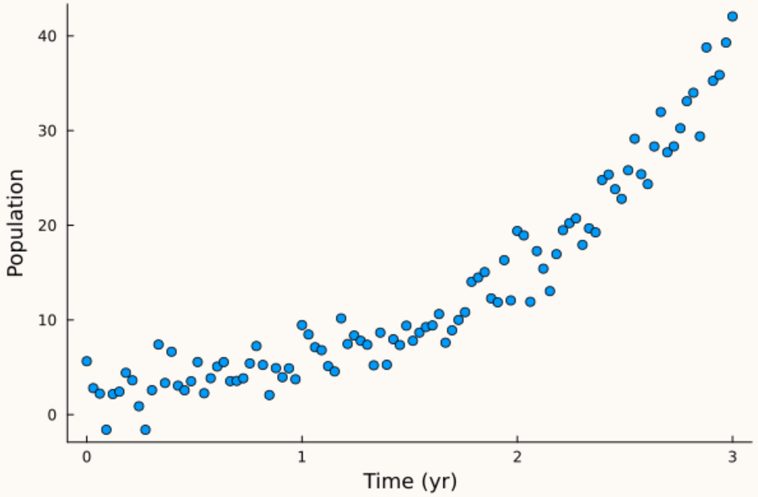
Example:

- Goal: find the growth of a population
- Data: Population statistics
- Model: Exponential

$$M(a,b) = \{t \mapsto a \cdot \exp(bt)\}$$



ELEMENTS OF AN INVERSE PROBLEM

MEASUREMENT OPERATOR

- A linear operator

$$\mathfrak{M}: \mathfrak{P} \rightarrow \mathfrak{D}$$

- $\mathfrak{P}$: Parameter space
- $\mathfrak{D}$: Data space

NOISE MODEL

- A noise map
  $n: \mathfrak{P} \times \mathfrak{D} \rightarrow \mathfrak{D}$

Accounting for

- Detector noise
- Modelling error

PRIOR MODEL

- Purpose: To reduce noise
- Restricts the parameter space $\mathfrak{P}$ based on prior information.

IMAGING PROBLEM:

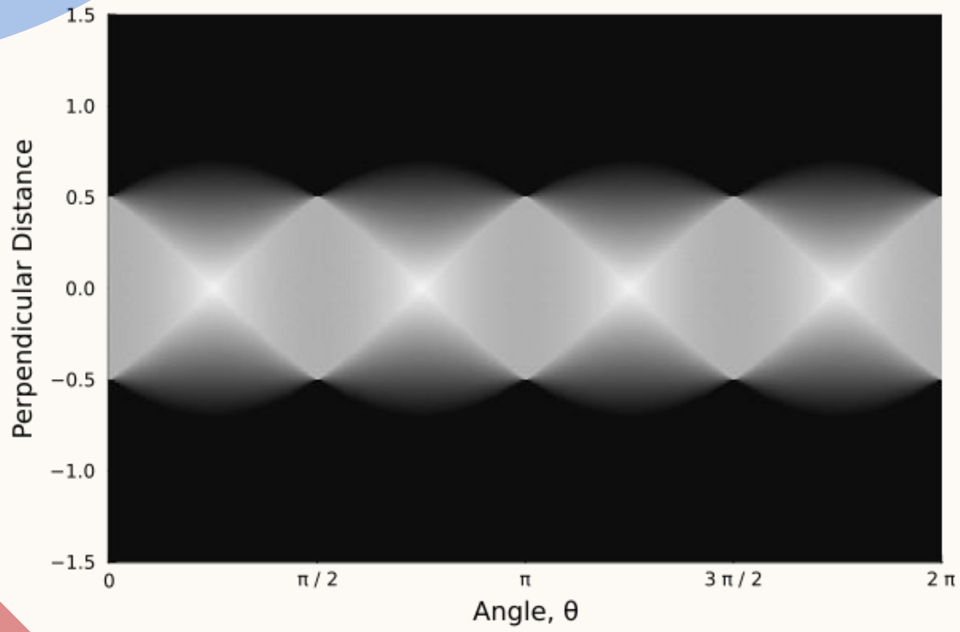


Fig. Sinogram of an unknown shape from x-ray data.

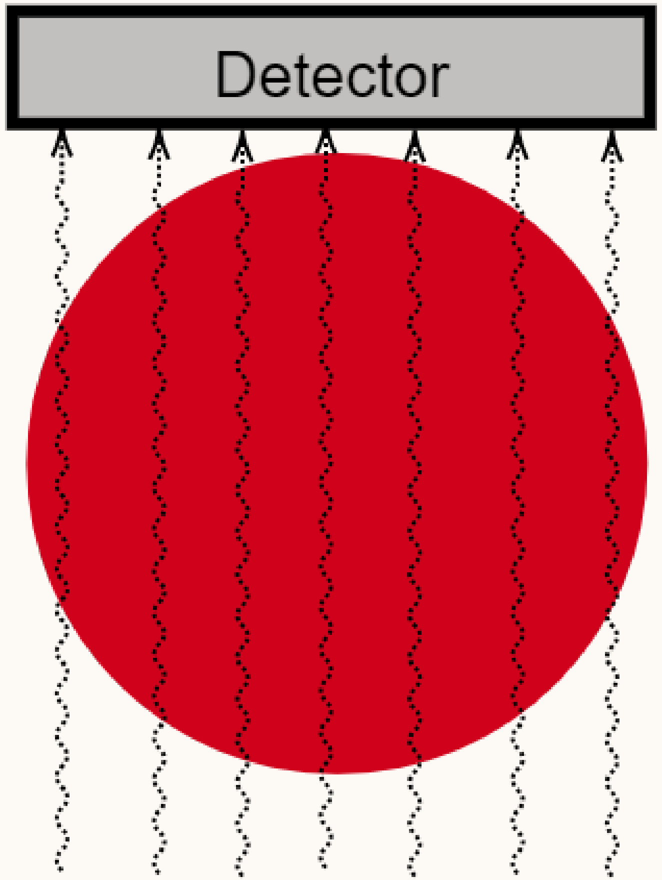


Fig. X-rays passing through an object and being picked up by a detector on the other side.

IMAGING PROBLEM:

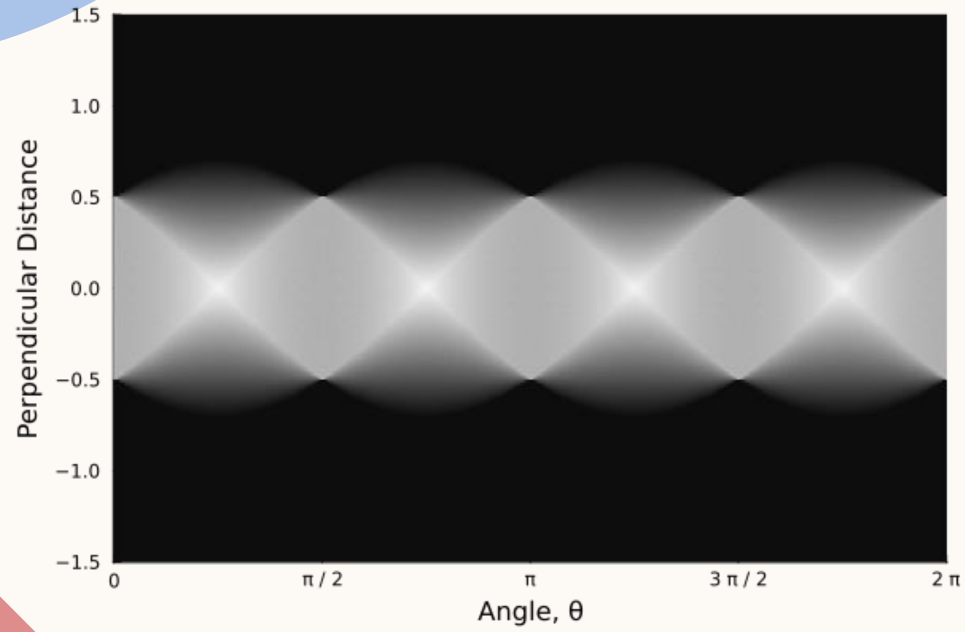


Fig. Sinogram of a square shape from x-ray data.

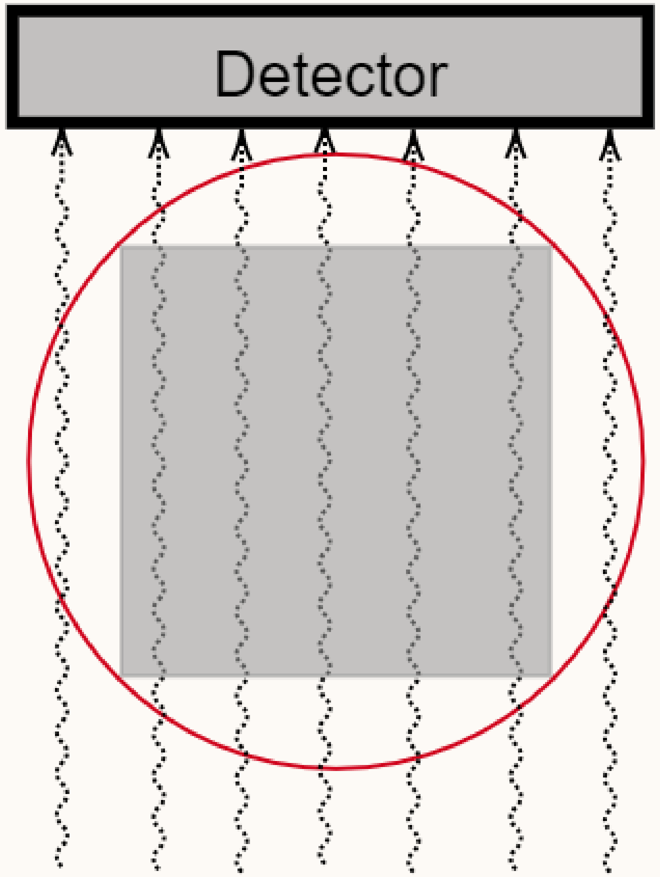


Fig. X-rays passing through a uniform square and being picked up by a detector on the other side.

RADON TRANSFORM

- The Radon Transform is a linear operator

$$R: L^2(\mathbb{R}^n) \longrightarrow L^2(S^{n-1} \times \mathbb{R})$$

defined by

$$Rf(\theta, s) = \int_{\langle \theta, x \rangle = s} f(x) dx$$

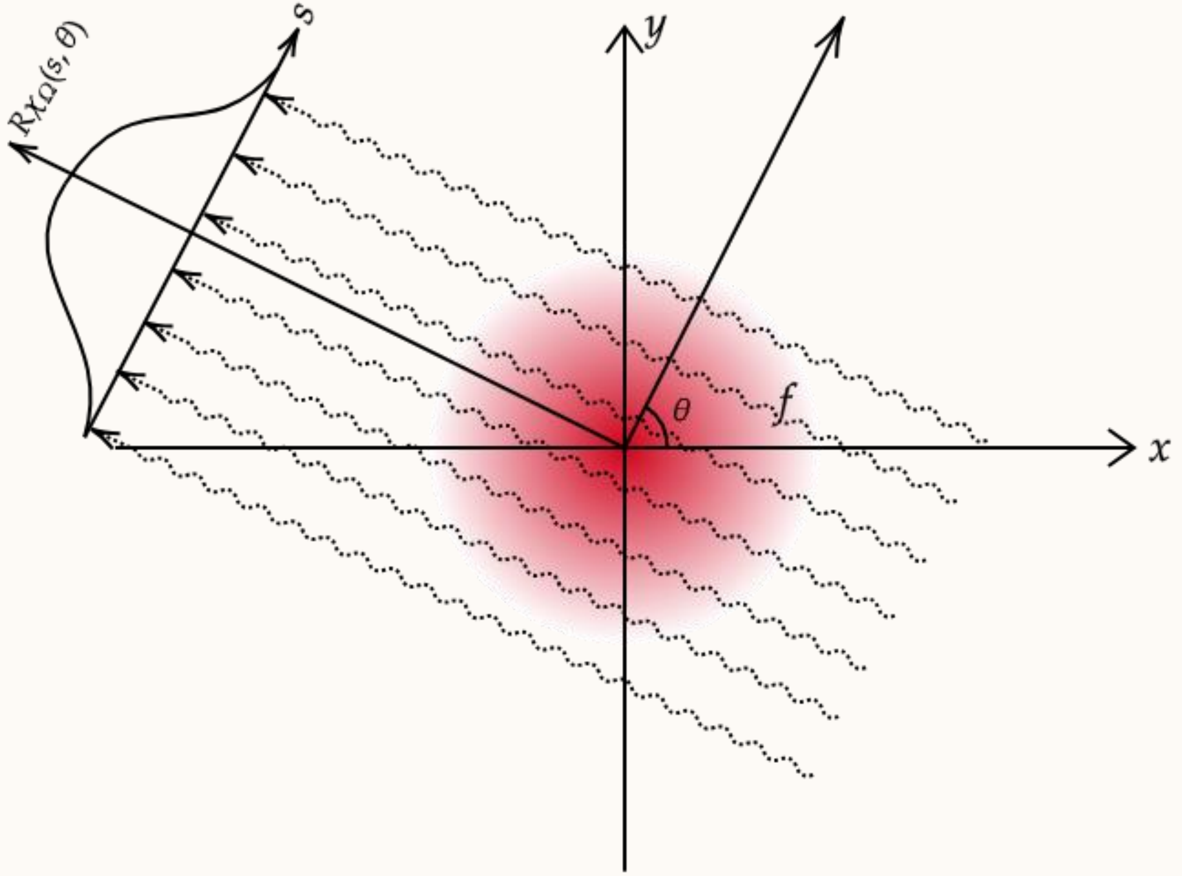
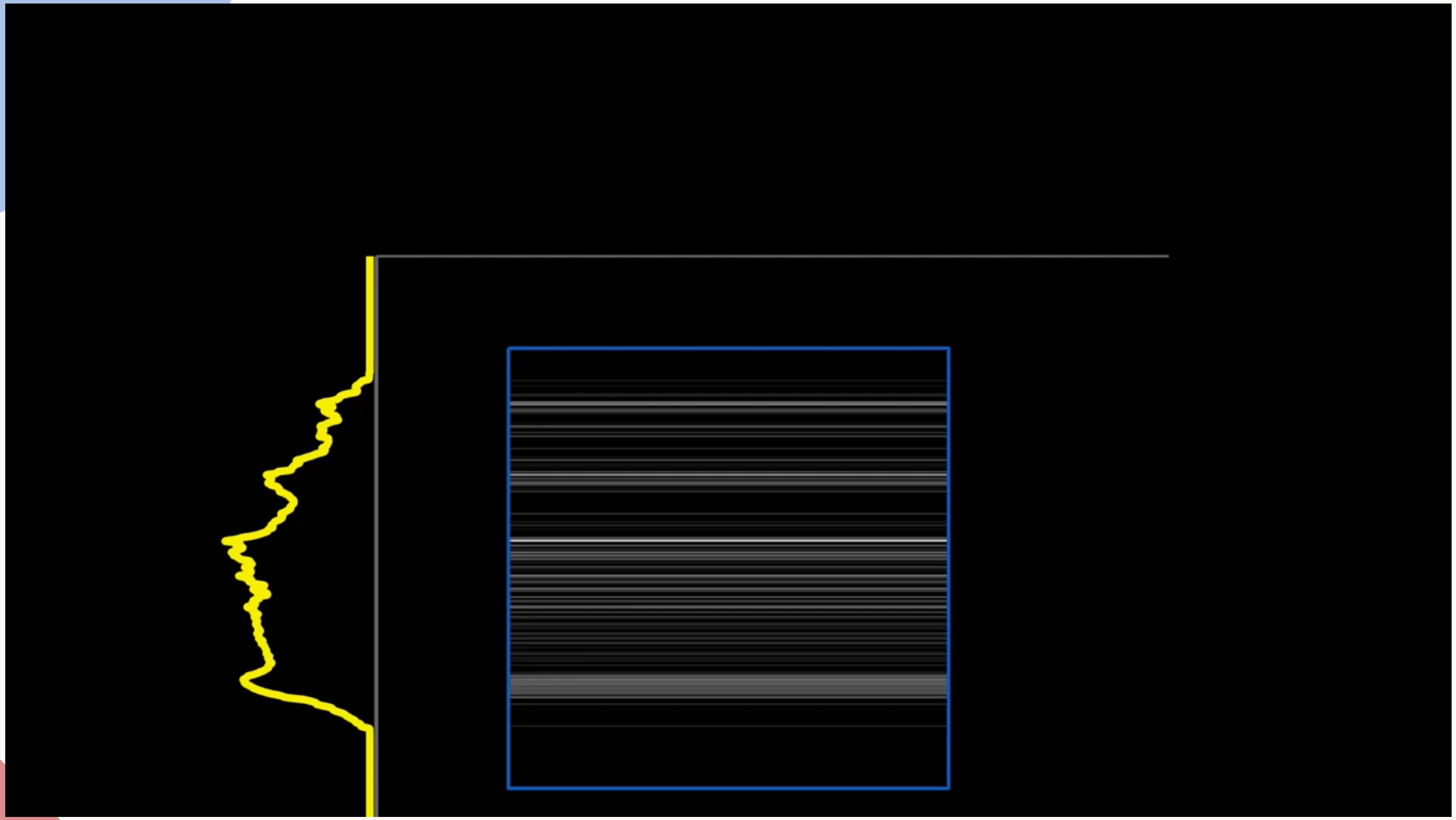


Fig. Model for Radon transform of a two-dimensional function f

IMAGING PROBLEM:



Vid. Samuli Siltanen Filtered Back Projection image reconstruction: https://www.youtube.com/@ssiltane/videos

RESTRICTED DATA TRANSFORM

Data:

- Ω: A bounded convex domain
- $R\chi_{\Omega}(\theta, 1)$: Tangent distance between boundary points

Goal:

- Arclength of $\partial\Omega$
- A parameterization $\gamma: S^1 \longrightarrow \partial\Omega$ in terms of the angle of tangency

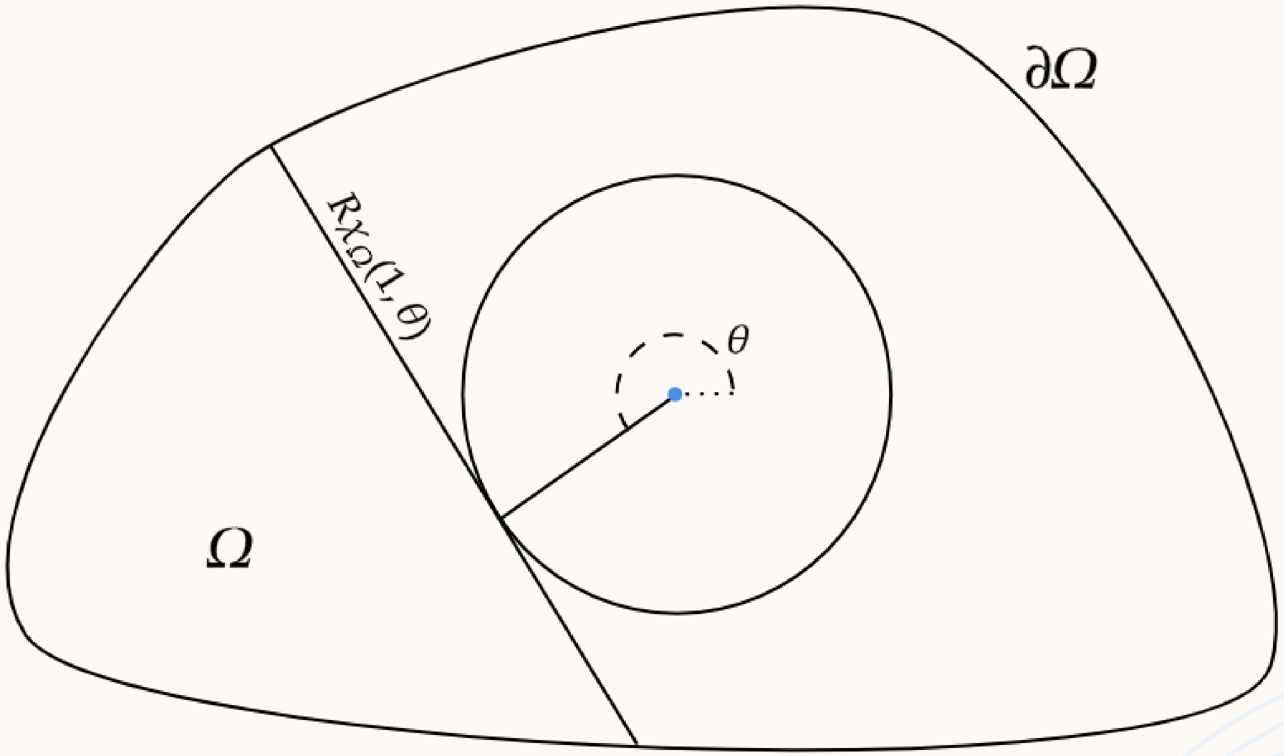


Fig. Model of a convex domain with interior circle and tangent data

RESTRICTED DATA TRANSFORM

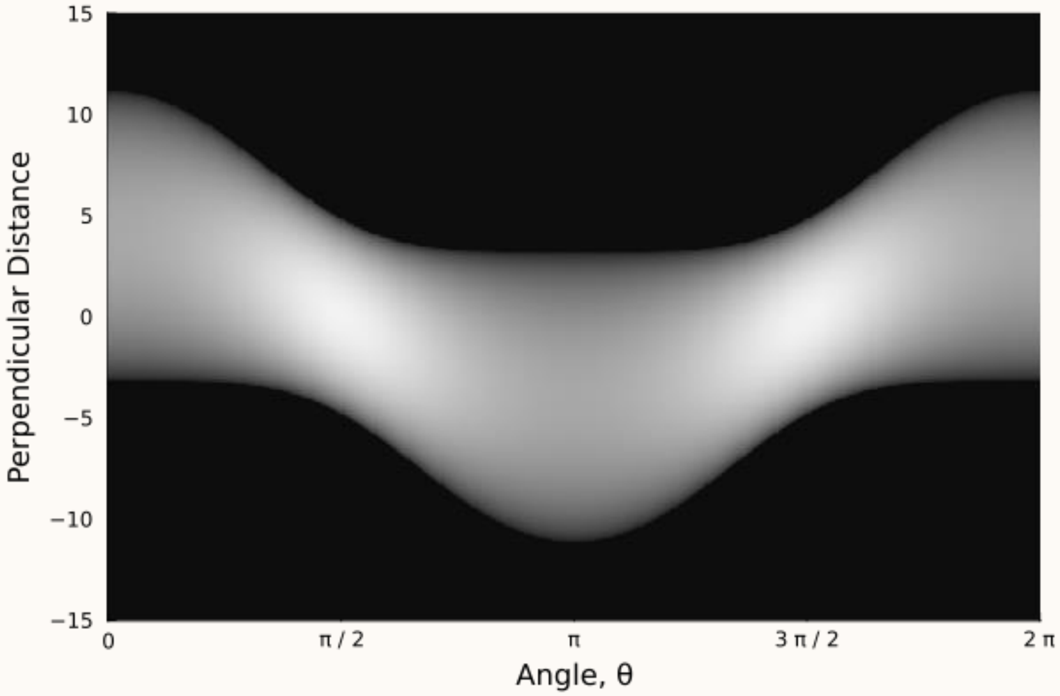


Fig. Sinogram of an unknown shape relative to an origin in its interior

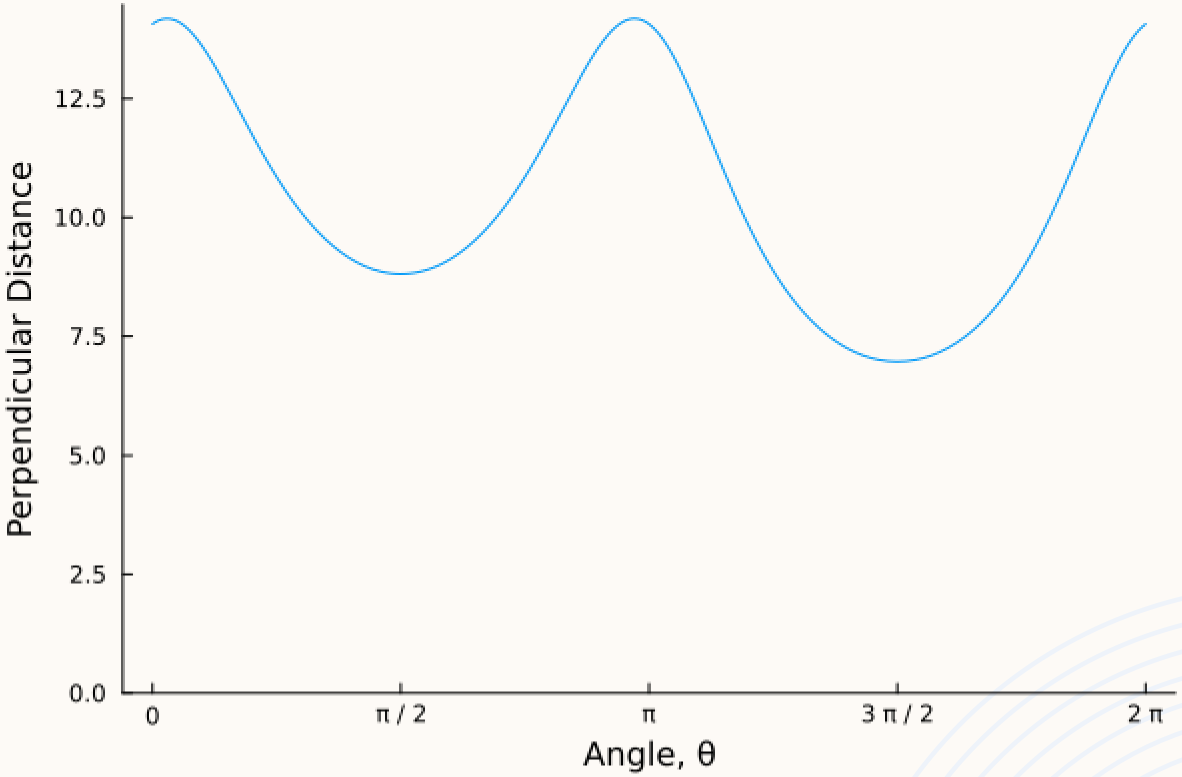


Fig. Sinogram slice at perpendicular distance 1.

RESTRICTED DATA TRANSFORM

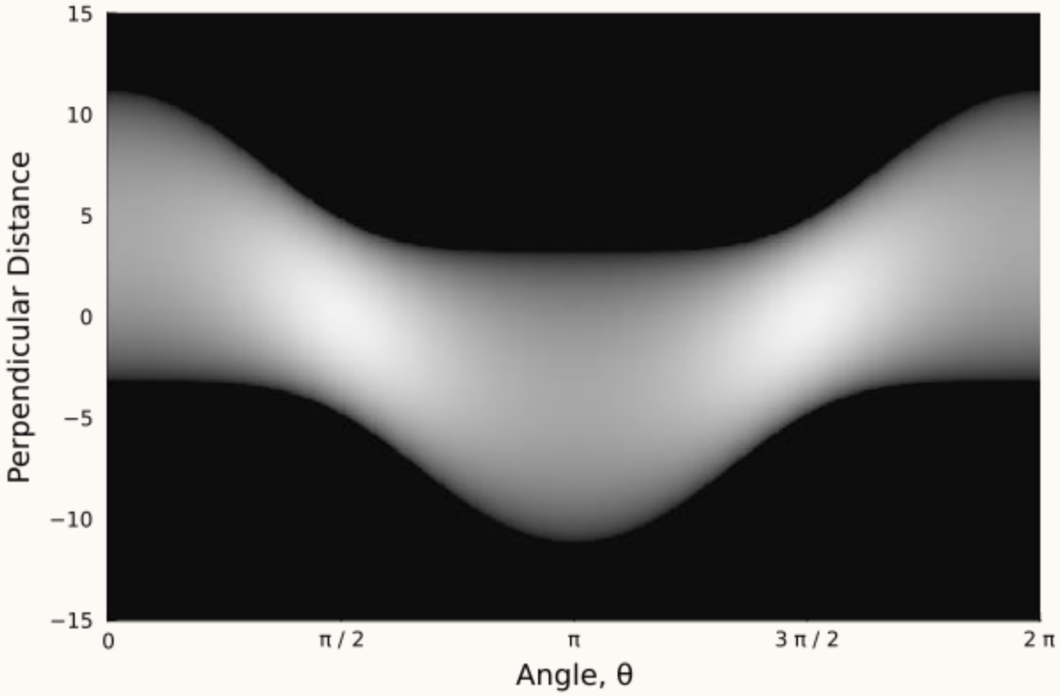
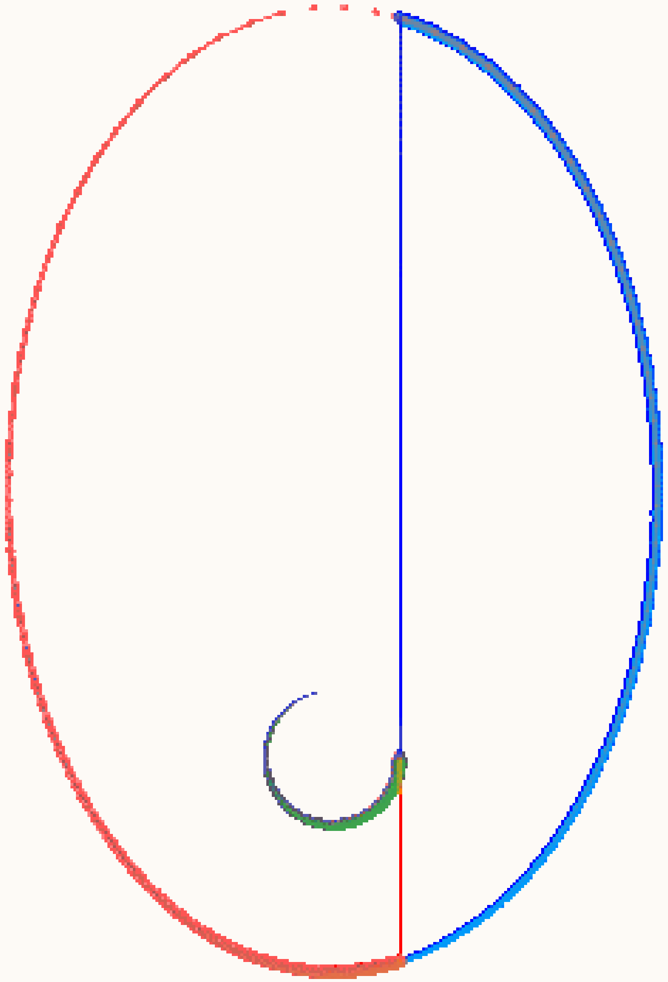


Fig. Sinogram of an ellipse relative to an origin in its interior



Gif. Ellipse reconstruction from to an interior circle

MICROLOCAL METHODS

Probing Singularities

DISTRIBUTIONS

Motivation:

- A domain Ω is determined by its boundary $\partial\Omega$
- As an $L^2(\mathbb{R}^2)$ function $\chi_{\partial\Omega}$ is undetectable
- However, $\chi_{\partial\Omega}$ can act on an $L^2(\mathbb{R}^2)$ function f through integration:

$$\langle \chi_{\partial\Omega}, f \rangle = \int_{\partial\Omega} f\,dx$$

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Definition:

A **distribution** χ on $\mathbb{R}^n$ is a continuous linear map from $C_c^\infty(\mathbb{R}^n)$ to $\mathbb{R}$.

- The space of distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$
- $L^2(\mathbb{R}^n)$ embeds into $\mathcal{D}'(\mathbb{R}^n)$ as $f \mapsto T_f$ with

$$T_f(\varphi) = \int f\varphi dx \, , \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

DISTRIBUTIONS

Properties:

- If $\chi \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha = (a_1, \dots, a_n)$ is a multi-index, $\partial_\alpha \chi$ is defined by
$$\langle \partial_\alpha \chi, \varphi \rangle = |-1|^{|\alpha|} \langle \chi, \partial_\alpha \varphi \rangle$$
- The Radon and Fourier transforms are defined for distributions χ via the formula
$$\langle R\chi, \varphi \rangle = \langle \chi, R^* \varphi \rangle \ and \ \langle \hat{\chi}, \varphi \rangle = \langle \chi, \hat{\varphi} \rangle$$
- If $\Phi: X \to Y$ is a diffeomorphism of open sets, and χ is a distribution on Y, then
$$\langle \Phi^* \chi, \varphi \rangle = \langle \chi, \varphi \circ \Phi |\det(D\Phi)| \rangle$$

Examples:

- All $L^2(\mathbb{R}^n)$ functions are distributions
- If Y is a submanifold of $\mathbb{R}^n$, its characteristic function χ_Y can be realized as a non-zero distribution
- The dirac delta “function” δ_0 can be realized as the distribution
$$\langle \delta_0, \varphi \rangle = \varphi(0), \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

SINGULARITIES AND WAVEFRONTS

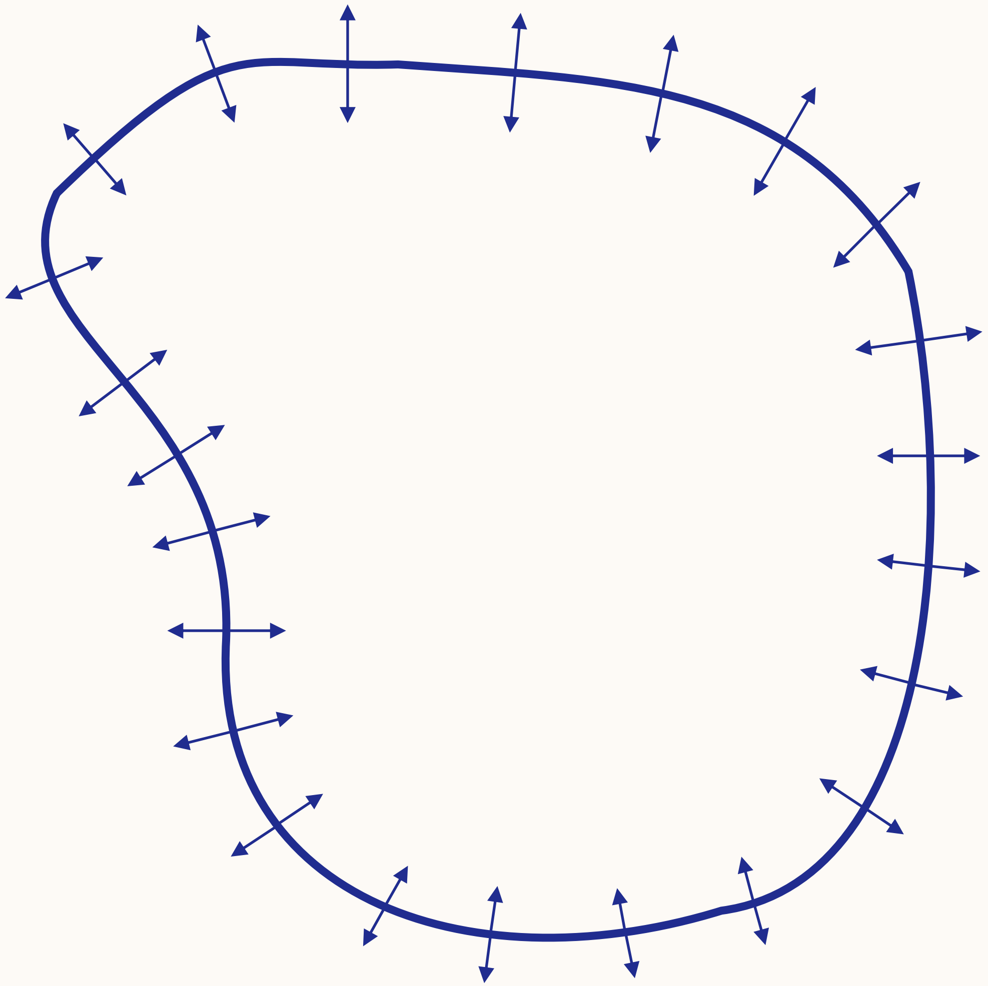
Definition:

Given $\chi \in \mathcal{D}'(X)$, the closed subset of X defined by

$$WF(\chi) \coloneqq \{(x, \xi) \in X \times \mathbb{R}^n : \xi \in \Sigma_x(\chi)\}$$

Is called the **wave front set** of χ, where $\Sigma_x(\chi)$ is the collection of "singular directions" of χ at x.

The projection $\pi_X: WF(\chi) \to X$ has as image the **singular support** of χ.

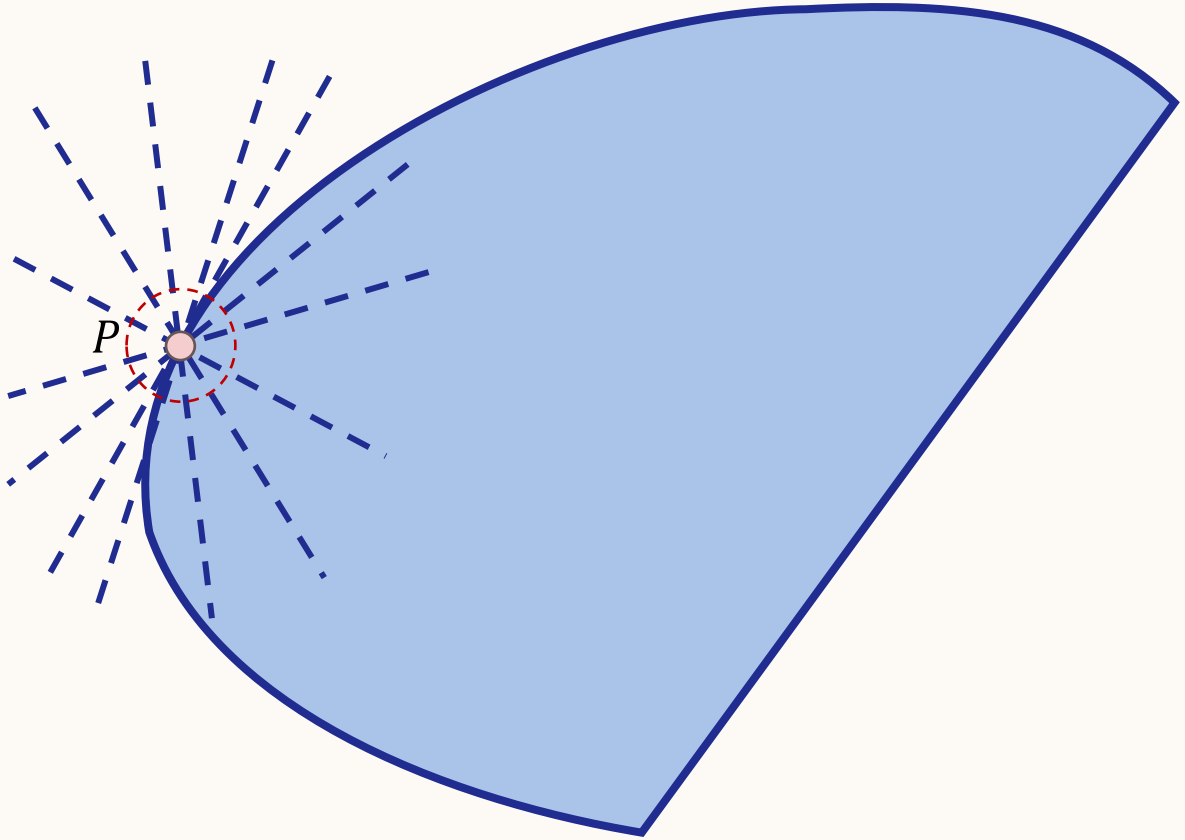


SINGULARITY DETECTION IN RADON TRANSFORM

Remark:

The Radon transform $R\chi$ of a distribution $\chi \in \mathcal{D}'(X)$ detects the singularities of χ:

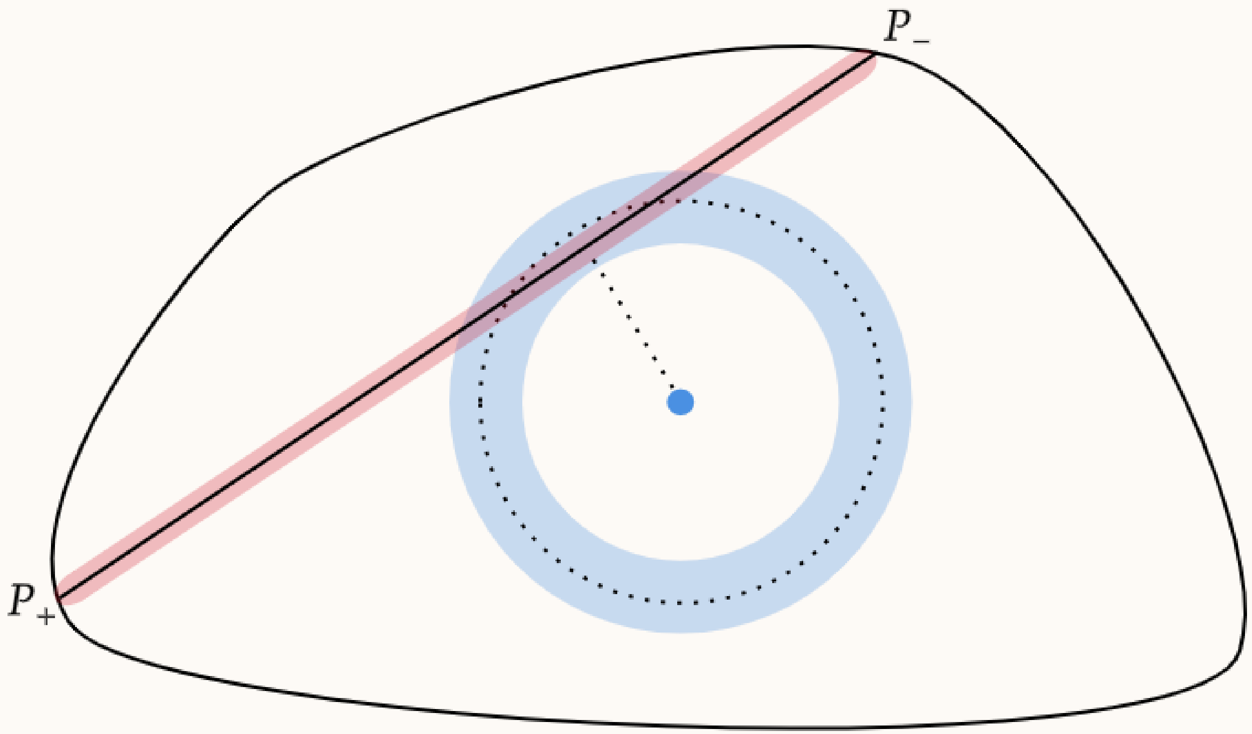
$$WF(R\chi) = \{(L,\eta) : \exists (x,\xi) \in WF(\chi), x \in L, \;\; \xi \perp L, \eta = \pm||\xi||dL\}$$



INFINITESIMAL THICKENING

Methods of microlocal analysis can still be used to determine equations on limited data

$$\partial_\theta R\chi_\Omega(1,\theta) = \langle (xv_y - yv_x)\delta_{\partial\Omega}\delta_L, 1\rangle = \frac{\det(P_+\ \nu(P_+))}{|\det(T(P_+)\ T_\theta)|} + \frac{\det(P_-\ \nu(P_-))}{|\det(T(P_-)\ T_\theta)|}$$



INFINITESIMAL THICKENING

On the other hand, using the thickened data can provide sufficient conditions for uniqueness

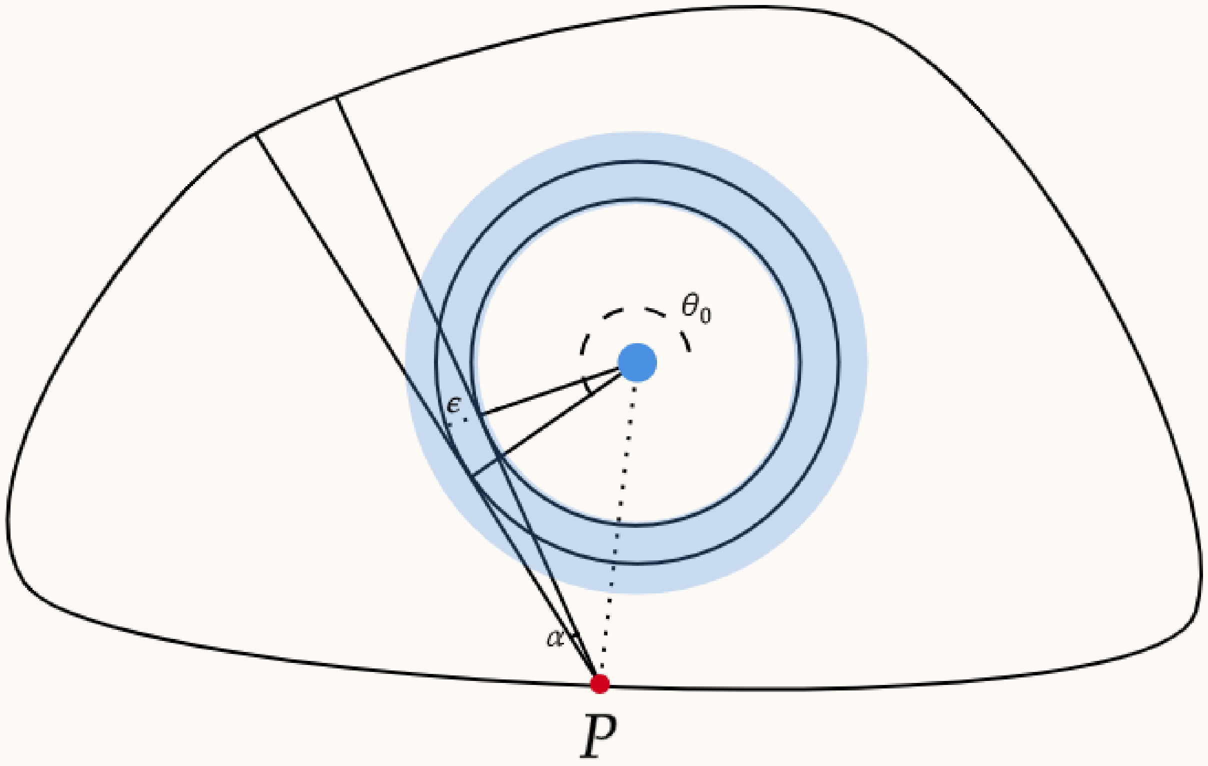


Fig. Local parameterizations from thickened data.

SUMMARY

- What are inverse problems
- The appearances of inverse problems in imaging and the Radon Transform
- Example of a limited data inverse problem
- Local techniques for studying inverse problems

THANK YOU FOR YOUR TIME!

Any Questions?